

# Quantum Gravity and Equivariant Cohomology\*

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## Abstract

A procedure for obtaining correlation function densities and wavefunctionals for quantum gravity from the Donaldson polynomial invariants of topological quantum field theories, is given. We illustrate how our procedure may be applied to three and four dimensional quantum gravity. Detailed expressions, derived from super-BF gauge theory, are given in the three dimensional case. A procedure for normalizing these wavefunctionals is proposed.

CTP # 2340  
hep-th/9407177

July 1994

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\*This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under cooperative agreement #DE-FC02-94ER40818.

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# 1 Introduction

Topological invariants on a manifold are a subset of diffeomorphism invariants. Thus we expect that elements of the set of topological invariants should be a subset of the quantum gravity observables. Additionally, it is generally believed that observables, which are elements of the BRST complex, may be used to construct vertex operators or wavefunctionals for the theory. Consequently, should we succeed in constructing observables for quantum gravity, we might also be able to construct wavefunctionals. These statements form the nexus for the present work. The puzzle is how to find representations of topological invariants in quantum gravity theories in sufficient generality so as not to explicitly exploit the topological nature of low-dimensional gravitational theories. In this paper, we will give a formal procedure for constructing operators which have the interpretation as the densities of correlation functions of observables and which lead to wavefunctionals, in this fashion.

Loop observables, which are constructed from Wilson loops, have been proposed [1, 2] for four dimensional canonical gravity in the Ashtekar formalism [3] via the loop representation [4]. In this way, observables which measure the areas of surfaces and volumes of regions have been constructed [2]. These are intricate constructions and we wonder if they may be placed in a different context via appealing to the geometry of the space of solutions to the constraints. From observables, we expect to be able to find states, and, perhaps, their wavefunctionals. Put into focus, our quest for a geometrical interpretation for general quantum gravity is a hope that we may be able to exploit the geometry to directly construct wavefunctionals. This is not to mean that we are diminishing the importance of observables.

Indeed, the geometry which underlies gauge field theories suggests another way of representing wavefunctionals; this will be the focal point of our exploration in this work. In particular, as both three dimensional gravity [5, 3] and the super-BF gauge theory [7] of flat  $SO(2, 1)$  or  $SO(3)$  connections (in which the geometry of the space of connections is explicit) share the same moduli space, these theories are natural choices for experimentation on this idea. We will find an interesting relation between the polynomial topological invariants of three dimensional flat connection bundles, which are the analogs of Donaldson's invariants [6] for self-dual connections in four dimen-

sional Yang-Mills gauge theory, and correlation densities of three dimensional quantum gravity. This does not mean that we will find correlation densities of new observables. We expect that the ones we will obtain may be decomposed in terms of Wilson loops. Further pursuit of our ideas then lead us to expressions for canonical and Hartle-Hawking wavefunctionals which satisfy the constraints of three dimension gravity. By exploiting previous work on four dimensional topological gravity, we are also able to sketch how our approach works in this physical dimension. Due to the fact that much more is known about the associated three dimensional topological quantum field theories (TQFT's) [7] than four dimensional topological gravity, we are presently unable to give expressions which are as detailed as those for three dimensional quantum gravity. We should point out that while the correlation densities and wavefunctionals which may be constructed via our approach for three dimensional gravity are likely to span the full space of such quantities, we do not expect this to be the case for four dimensional gravity. The reason is simply that the phase space of the former theory and of TQFT's is finite dimensional while that of the latter is not.

Our work is relatively formal as our objective is to establish an approach to solving some of these long standing problems of quantum gravity. In particular, we give expressions in terms of path integrals which, in principle, may be computed exactly. These path integrals appear as those of topological quantum field theories which are strongly believed to be, at worst, renormalizable [7]. This allows us to make use of BRST analysis techniques in order to establish our results. A related approach for the computation of scattering amplitudes in string theory was undertaken by one of us in ref. [8].

Commencing, we establish the framework of our approach while attempting to be as general as possible, in the next section. Implementation of the approach is carried out for three dimensional BF-gauge theory, in general, and 3D quantum gravity, in particular, in section 3. Expressions for correlation densities are given in sub-section 3.1 while wavefunctionals may be found in sub-section 3.2. The four dimensional case is sketched in section 4. Our conclusions may be found following that section. In addition, appendices summarizing BF-gauge theories and super-BF gauge theories are given. In appendix C we suggest the possible existence of polynomial invariants in pure three quantum gravity, before applying our approach. Our global notations are given in appendix D.

## 2 The Heuristic Construction

As was discussed in the introduction, our approach is to first find correlation densities and then extract the wavefunctionals from them. Thus, in this section, we first concentrate on our general approach to obtaining the correlation densities. Then, we will discuss how to obtain the wavefunctionals from them, at the end of this section.

### 2.1 Correlation Densities

Given a field theory, one is interested in its physical states and the observables; *i.e.*, functionals and functions of the fields which obey the constraint of the theory. One reason why observables are important is that from them physical correlation functions can be constructed. However, it is not necessary to find observables in order to construct physical correlation functions of the fields. As an example, given a function,  $\hat{\mathcal{O}}$ , of the fields, we will only demand that the vacuum expectation value  $\frac{\delta \langle \hat{\mathcal{O}} \rangle}{\delta g_{\mu\nu}}$  vanishes, where  $g_{\mu\nu}$  is some background metric. This allows for  $\frac{\delta \hat{\mathcal{O}}}{\delta g_{\mu\nu}} \neq 0$ . The  $\hat{\mathcal{O}}_i$  we will construct will have the property that generally  $\frac{\delta \langle \prod_{i=1}^n \hat{\mathcal{O}} \rangle}{\delta g_{\mu\nu}} \neq 0$  for  $n \geq 2$ . Thus they are really physical correlation function densities. In this section, we will describe how we can use TQFT's in order to construct, a set of the  $\hat{\mathcal{O}}$ 's for a general field theory (GFT). Our focus will be on quantum gravity for which topological observables are of interest.

Take a GFT for fields,  $X$ , which are sections of a bundle over a manifold,  $M$ , and whose space of physical fields is called  $\mathcal{N}$ . Construct [9] a TQFT which describes the geometry of a subspace of  $\mathcal{N}$ , which we call  $\mathcal{M}$  (the dimension of  $\mathcal{M}$  is finite). In this way, we have projected the GFT onto the TQFT. Expectation values of observables in the TQFT (which we generally know how to write), are topological invariants of  $M$ . Now if the TQFT has the constraints,  $\mathcal{G}$ , of the GFT as a subset of it's own constraints then we can construct physical correlation functions and wavefunctionals of the GFT, with the use of the TQFT. We now describe two different ways of doing this.

First, suppose we are given a particular GFT for fields  $X$  and are able to construct a TQFT with fields  $X$  and  $Y$ . Let us require that this TQFT

has the same Lagrangian as the GFT plus additional terms which are also invariant under the local symmetries of the GFT<sup>1</sup>. Furthermore, we require that a subalgebra of the constraints of our TQFT is isomorphic to the constraint algebra of the GFT. In particular, the action of this subset on the  $X$ , in the TQFT is the same as the action of the GFT's constraints on  $X$ . As an example, take the GFT to be BF-gauge theory and the TQFT to be super-BF gauge theory.

Now take a set of observables in the TQFT and *almost* compute their correlation function. By this we mean the following. Integrate the path integral over all the fields that are present in the TQFT but not in the GFT; that is, over  $Y$ . We then get an expression (which is typically non-local),  $\hat{\mathcal{O}}$ , in terms of the fields  $X$ . The expectation value of  $\hat{\mathcal{O}}$ , in the GFT, is a topological invariant of  $M$ . More precisely,

$$\langle \hat{\mathcal{O}} \rangle_{GFT} = \int [dX] e^{-S_{GFT}} \hat{\mathcal{O}}(X) = \int [dX][dY] e^{-S_{TQFT}} \mathcal{O}(X, Y) \quad , \quad (2.1.1)$$

where  $\mathcal{O}$  is a product of observables in the TQFT,  $S_{GFT}$  is the action of the GFT and  $\hat{\mathcal{O}}$  is a gauge invariant, non-local expression in terms of the original fields. Really what we are doing is taking the original theory and coupling special “matter” to it, and using the matter part to construct physical correlation functions. However, using TQFT's has additional rewards. First, these expressions are computable as the theories are, at worst, renormalizable. Second, we will see that we will be able to write expressions without integrating over the entire spacetime manifold, which satisfy the constraints of quantum gravity.

Although it is tempting to call the correlation densities observables, this can only be done with qualification as they do not have one of the important properties we associate with observables. That is, generally the product of two or more of them is not an observables in the sense that this product's vacuum expectation value will not be diffeomorphism invariant.

To place the above arguments in a geometrical setting, let us look at the geometry of the space of connections [10]. This argument applies, in principle, to any gauge theory built from a Yang-Mills fields space. Let  $P(M, G)$  be a  $G$ -bundle over the spacetime manifold,  $M$ , and  $\mathcal{A}$  be the space of its connections. Forms on the space  $P \times \mathcal{A}$ ,  $\Omega^{(p,q)}(P \times \mathcal{A})$ , will be bi-graded

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<sup>1</sup> $Y$  may be thought of as the supersymmetric partners of  $X$  and the additional terms in the action as supersymmetric completion.

inheriting degrees  $p$  from  $M$  and  $q$  from  $\mathcal{A}$ . A connection  $\mathbf{A} \equiv A + c$  may be introduced on the bundle  $P \times \mathcal{A}$  along with an exterior derivative  $\mathbf{d} \equiv d + Q$  where  $d(Q)$  is the exterior derivative on  $M(\mathcal{A})$ . The object,  $c$  is the ghost field of the Yang-Mills gauge theory. The total form degree of  $\mathbf{A}$  is one and is given by the sum of the degree on  $M$  and ghost number. The curvature of the connection  $\mathbf{A}$  is  $\mathbf{F} = \mathbf{d}\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}] = F + \psi + \phi$ , where the  $(2, 0)$  form  $F = dA + A \wedge A$  is the usual curvature of  $P$ ,  $\psi = QA + d_A c$  is a  $(1, 1)$  form and  $\phi = Qc + \frac{1}{2}[c, c]$  is a  $(0, 2)$  form. Gauge invariant and metric independent operators may be constructed out of these objects. They are the Donaldson invariants written in a field theoretic language. Thus we will be attempting to recover these geometrical objects which already exist, but are hidden, in physical gauge theories.

## 2.2 Wavefunctionals

We can also obtain wavefunctionals of the GFT's fields,  $X$ , which satisfy the latter theories constraints. A general method will be described first, then another prescription which we will later see works for three dimensional gravity, but which is not guaranteed to work in general, will be given. In the following we will use the term geometrical sector to refer to those fields which are realized as the curvature components for the geometry of the universal bundle over  $X$ . For example, these would be  $(A, \psi, \phi)$  in a theory defined over a Yang-Mills field space.

First, take the TQFT to be defined over a spacetime manifold  $M$  with a boundary,  $\partial M$ , which is homeomorphic to the surface,  $\Sigma$ , we wish to quantize the GFT on. As for the GFT, let the phase space of the TQFT be even-dimensional. Note that  $M$  need not be diffeomorphic to  $\Sigma \times \mathbb{R}$ . Form the correlation function of a set of observables in this TQFT. Choose a polarization and functionally integrate over the  $X$  and  $Y$  sets of fields in the TQFT with boundary conditions on  $\Sigma$ . Then, the correlation function will yield a functional of the boundary values of half of the Cauchy data for the  $X$  fields, call that set  $X|_{\Sigma}$ , and half of the  $Y$  fields, call that set  $Y|_{\Sigma}$ . By construction this is a Hartle-Hawking wavefunctional for the TQFT which is guaranteed to be computable since, at worst, TQFT's are renormalizable:

$$\Psi[X|_{\Sigma}, Y|_{\Sigma}] = \int [dX][dY] e^{-S_{TQFT}} \mathcal{O}(X, Y) \quad . \quad (2.2.1)$$

Here  $S_{TQFT}$  is the TQFT action on the manifold with boundary,  $\Sigma$ . The wavefunctional,  $\Psi[X|_{\Sigma}, Y|_{\Sigma}]$  is diffeomorphism invariant due to the properties of TQFT's. For the particular TQFT, any fields which appear in  $\Psi[X|_{\Sigma}, Y|_{\Sigma}]$  and which are not in the geometrical sector, should be integrated out. Then all the  $Y$  fields which remain in  $\Psi[X|_{\Sigma}, Y|_{\Sigma}]$  may be replaced by non-local expressions involving  $X$  and  $\frac{\partial X}{\partial m}$ . This is an idiosyncrasy of TQFT's. Then since  $\frac{\partial X}{\partial m}$  is a function of  $X$ , we obtain

$$\Psi[X|_{\Sigma}, Y|_{\Sigma}] \implies \Psi[X|_{\Sigma}] \quad . \quad (2.2.2)$$

In practice, we find that those  $Y|_{\Sigma}$  fields which appear in  $\Psi[X|_{\Sigma}, Y|_{\Sigma}]$  are Grassmann-odd and the projection to  $\Psi[X|_{\Sigma}]$  stated above is performed by first choosing a basis for  $T^*\mathcal{M}$ , expanding those  $Y|_{\Sigma}$  in this basis and then expanding  $\Psi[X|_{\Sigma}, Y|_{\Sigma}]$  as a superfield whose components are wavefunctionals,  $\Psi[X|_{\Sigma}]$ .

This approach leads us to the following ansatz for a normalization procedure which stems from the axiomatic approach [11] to TQFT's. Given two wavefunctionals,  $\Psi_1$  and  $\Psi_2$ , defined on diffeomorphic boundaries,  $\partial M_1$  and  $\partial M_2$ , we might try defining the inner product by gluing the two manifolds together. This will result in a path integral of some observable of the TQFT defined on the glued manifold. As these expressions are finite this gives a possible normalization procedure. We defer the exact construction to future work [12].

A second approach to constructing the wavefunctionals stems from the observation that, in the above, we took the wavefunctionals of the TQFT and projected onto the  $X$  subspace to obtain the wavefunctionals of the GFT. Thus it is suggestive to simply construct the wavefunctionals of the TQFT by any means possible and then apply the projection. Thus we need not restrict ourselves to Hartle-Hawking wavefunctionals but might also consider those obtained by directly analyzing the constraints of the canonically quantized TQFT.

Now let us specialize to a certain set of GFT's. For certain theories, such as three dimensional gravity, we may construct such wavefunctionals by building TQFT's which are defined in a background which solves the constraints of the GFT. We will call such TQFT's, servant theories. That is, in the GFT, we solve the constraints first and then quantize. The quantization then demands that we find wavefunctionals which have support only on the

constraints' solutions. Realizing this, we construct correlation functions in a servant TQFT which is defined over a certain background. As we will see in the next section, this works when the servant TFT is of the Schwarz [13] type. As the servant TFT's must be topological, this approach restricts the background; *i. e.*, those  $X|_{\Sigma}$  which solve the constraints, to be non-propagating fields or global data. Thus we expect that this approach will only work for certain sectors of four dimensional gravity.

Having given a cursory discussion of our procedures for obtaining observables and wavefunctionals, let us now turn to some specific applications. Three dimensional quantum gravity and BF-gauge theories, in general, are first.

### 3 Application to 3D BF Gauge Theories

As BF-gauge theories are TFT's, they are the logical choice for the first application of the ideas discussed in the previous section. Although our analysis below may be carried out in arbitrary dimensions, we will focus on 2+1 dimensional manifolds. In this dimension, BF-gauge theories are of more than a passing interest; as with gauge group  $G = SO(2, 1)$ , they are known [5, 3, 14] to be theories of quantum gravity. In subsection 3.1, we will study the construction of correlation densities in the covariant quantization of BF-gauge theories based on the geometry of the universal bundle. Then in subsection 3.2 we will give formal expressions for canonical and Hartle-Hawking wavefunctionals of BF-gauge theories again based on the geometry of the universal bundle. Where appropriate, we will make allusions to three dimensional quantum gravity

Before proceeding we would like to be further explain the rationale for choosing BF-gauge theories (see appendix A) as a first application of our constructions. There are cohomological field theories (or TQFT's), called super-BF gauge theories, which share the same moduli space. As quantum field theories, they are very closely related [15] and the manifest appearance of the geometry of the constraint space of BF-gauge theories in the super-BF gauge theories will be most useful. These two facets make the construction of observables and wavefunctionals for BF-gauge theories from super-BF gauge



theories highly suggestive and, as we will find, possible.

### 3.1 Pulling Back $H^*(\mathcal{M})$ to BF-gauge theories

Define  $\mathcal{N}$  to be the restriction of  $\mathcal{A}$  to flat connections:  $\mathcal{N} \leftrightarrow \mathcal{A}|_{F=0}$  and  $\mathcal{M} = \mathcal{N}/G$  to be the moduli space of flat connections. Let  $m^I$ ,  $I = 1, \dots, \dim \mathcal{M}$  be local coordinates on  $\mathcal{M}$ . Flat connections are then parameterized as  $A(m)$ . Given two nearby flat connections as  $A(m)$  and  $A(m+dm)$ , we expand the latter to see that the condition for it to also be a flat connection is that

$$d_A \frac{\partial A}{\partial m^I} dm^I = 0 . \quad (3.1.1)$$

By definition, the zero-mode of the  $(1,1)$  curvature component on  $P \times \mathcal{A}$ ,  $\psi^{(0)}$ , satisfies the equation

$$d_A \psi^{(0)} = 0 , \quad (3.1.2)$$

where  $A$  is a flat connection. Thus we immediately find a basis from which  $\psi^{(0)} = \psi_I^{(0)} dm^I$  may be constructed; namely,  $\psi_I^{(0)} = \frac{\partial A}{\partial m^I}$ .

We seek observables in the BF-gauge theory which we can formally write in terms of  $\frac{\partial A}{\partial m^I}$  assuming we have chosen a coordinate patch on  $\mathcal{N}$ . In order for them to be observables they must be gauge invariant and diffeomorphism invariant. These conditions are related as we will soon see. Let us now turn to their construction.

For a homology two-cycle,  $\Gamma$ , on  $M$ , we define

$$\mathcal{O}_{IJ} \equiv \frac{1}{2} \int_{\Gamma} Tr \left( \frac{\partial A}{\partial m^I} \wedge \frac{\partial A}{\partial m^J} \right) . \quad (3.1.3)$$

Under a gauge transformation,  $A \rightarrow A^g$ ,  $\frac{\partial A}{\partial m^I}$  transforms into  $\frac{\partial A^g}{\partial m^I}$ . Then for an infinitesimal gauge transformation, with parameter  $\epsilon$ ,

$$\delta_{\epsilon} \mathcal{O}_{IJ} = \int_{\Gamma} Tr \left( \frac{\partial \epsilon}{\partial m^{[I}} \frac{\partial F}{\partial m^{J]}} \right) . \quad (3.1.4)$$

Thus we see that  $\mathcal{O}_{IJ}$  is gauge invariant if  $A$  is a flat connection. Hence it is a possible observable in BF-gauge theories.

A check of diffeomorphism invariance remains to be done. Diffeomorphisms of the manifold,  $M$ , by the vector field,  $K$ , are generated by the Lie derivative  $\mathcal{L}_K = di_K + i_K d$ . By direct computation,

$$\mathcal{L}_K \frac{\partial A}{\partial m^I} = i_K d_A \frac{\partial A}{\partial m^I} + \left[ \frac{\partial A}{\partial m^I}, \alpha(K) \right] + d_A \left( i_K \frac{\partial A}{\partial m^I} \right), \quad (3.1.5)$$

where  $\alpha(K) \equiv i_K A$ . If  $A$  is a flat connection, the first term in the right-hand-side of this expression vanishes. The second term is a gauge transformation. Although the last term is inhomogeneous the fact that it is a total derivative means that after integrating by parts and imposing the flat connection condition, its contribution vanishes. It then follows that  $\mathcal{O}_{IJ}$  is an example of a gauge invariant operator whose correlation functions in the BF-gauge theory are diffeomorphism invariants.

Having convinced ourselves, by the example above, of the existence of operators in BF-gauge theories which lead to diffeomorphism invariant correlation functions, we must now establish a procedure for constructing such quantities. This will be done by implementing the ideas in section 2; namely, almost compute the topological invariants from the super-BF gauge theory theory. By almost, we mean integrating over all of the fields in the functional integrals except for the gauge connection and the field  $B$ . This will leave us with a functional integral expression over the space of fields in the ordinary  $BF$  theory but with operator insertions at various points on the manifold. Now we know that we are in fact computing topological invariants. Then it follows that these operators, which will appear as functionals of the connection will be physical correlation densities<sup>2</sup> in the BF-gauge theory whose expectation values are topological invariants.

To illustrate the procedure, let us write the generic Donaldson polynomials as  $\mathcal{O}_i(\phi, \psi, F; \mathcal{C}_i)$  where  $\mathcal{C}_i$  is the cycle the observable is integrated over. Then we have to compute

$$\begin{aligned} \left\langle \prod_i \mathcal{O}_i(\phi, \psi, F; \mathcal{C}_i) \right\rangle_{SBF} &= (Z_{SBF})^{-1} \int [d\mu]_{SBF} e^{-S_{SBF}} \times \\ &\times \prod_i \mathcal{O}_i(\phi, \psi, F; \mathcal{C}_i), \end{aligned} \quad (3.1.6)$$

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<sup>2</sup>The relation of these quantities to what we normally expect observables to be is discussed in section 2.

where  $Z_{SBF}$  is the partition function of the super- $BF$  theory and  $[d\mu]_{SBF}$  (see appendices B and D) is the measure for the path integral over the fields  $\chi, \psi$ , etc. with the  $\chi$  zero-modes inserted. It is known (see appendix B) that certain classes of operators  $\mathcal{O}_i$  exist for which these correlation functions are topological invariants.

The integral over  $\lambda$  may be performed leading to the delta function  $\delta(\Delta_A \phi + [\psi, \star \psi])$ . This means that the  $\phi^a$  field is replaced by  $\langle \phi^a(x) \rangle = -\int_M G^{ab}(x, y)[\psi, \star \psi]_b$ , where  $G^{ab}(x, y)$  is the Green's function of the scalar covariant laplacian  $(\Delta_A)$ , in the computation of the correlation function of the observables. Since the functional integral has support only on flat connections, if there are no  $B$ -fields in the observables (as is the case for our  $\mathcal{O}$ 's), the correlation densities reduce effectively to functions of  $\psi$  and the flat connections only. In order for correlation functions to be non-zero, the product of the observables – reduced in this way – must include all  $\psi$  zero-modes. For a genus  $g \geq 2$  handlebody, this number is  $(g - 1) \dim(G)$ . Let us now look at the various classes of correlation functions.

The vacuum expectation value of a single  $\mathcal{O}$  is a topological invariant in the super-BF gauge theory. Hence, the gauge invariant operator ( $S_S$  is defined in appendix B)

$$\hat{\mathcal{O}}(A; \mathcal{C}) \equiv \int [d\mu]_S e^{-S_S} \mathcal{O}(\phi, \psi, F; \mathcal{C}) \quad , \quad (3.1.7)$$

has the property that its vacuum expectation value in the BF-gauge theory is a topological invariant. It is important to note that in general,  $\hat{\mathcal{O}}$  depends on the background metric on  $M$ . Furthermore, although it is gauge invariant, it is not in the cohomology of the,  $Q_{SBF} = Q^H + Q_{YM}$ , total BRST charge (see appendix B for a discussion on  $Q^H$ ), where  $Q_{YM}$  is the Yang-Mills BRST charge. Hence, its correlation functions will not be independent of the background metric, in general. Additionally, the  $\hat{\mathcal{O}}$  will be non-local operators in general. Although these last two points may be viewed as drawbacks of this approach, there is one important lesson to be learned here. This construction clearly demonstrated that the three dimensional analogs of Donaldson invariants give rise to operators in the BF-gauge theories whose vacuum expectation values, in the latter theories, are themselves topological and are physical in three dimensional quantum gravity. It should also be noted that although the fields  $B, c, \bar{c}, c', \bar{c}'$  appear in  $S_S$ , they do not survive the  $[d\mu]_S$  integration due to  $\psi$  zero-mode saturation.

Until this point, we have only looked at the vacuum expectation values of the  $\hat{\mathcal{O}}$ 's. Now, we would like to investigate the expectation value of  $\hat{\mathcal{O}}$  in any physical state of the BF-gauge theory. In particular, we would like to see whether or not such an expression is independent of the background metric,  $g_{\mu\nu}$ , used in forming the gauge fixed action. Let us suppose that such a state may be constructed out of the action of Wilson loop operators on the vacuum. Alternatively, we can ask whether or not the correlation function of the  $\hat{\mathcal{O}}$ 's with Wilson loop operators,  $W[R, \gamma] = \text{Tr}_R P \exp(\oint_\gamma A)$ , is background metric dependent. Hence we are led to study the functional integral

$$\begin{aligned}\mathcal{E}(R, \gamma, \mathcal{C}) &= \int [d\mu]_{SBF} e^{-S_{SBF}} \mathcal{O}(\phi, \psi, F; \mathcal{C}) W[R, \gamma] \\ &= \int [d\mu]_{BF} e^{-S_{BF}} \hat{\mathcal{O}}(A; \mathcal{C}) W[R, \gamma] \quad .\end{aligned}\quad (3.1.8)$$

Functionally differentiating  $\mathcal{E}(R, \gamma, \mathcal{C})$  with respect to the inverse metric,  $g^{\mu\nu}$ , we find

$$\frac{\delta \mathcal{E}(R, \gamma, \mathcal{C})}{\delta g^{\mu\nu}} = \int [d\mu]_{SBF} e^{-S_{SBF}} \Lambda_{\mu\nu} \mathcal{O}(\phi, \psi, F; \mathcal{C}) \text{Tr}_R P \left( \oint_\gamma \psi e^{\oint_\gamma A} \right) \quad , \quad (3.1.9)$$

after use of the properties of  $S_{SBF}$  and where  $\frac{\delta S_{SBF}}{\delta g^{\mu\nu}} = \{Q, \Lambda_{\mu\nu}\}$  with

$$\Lambda_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}} \int_M (\lambda d_A^* \psi + \lambda' d_A^* \chi + \bar{c}' d_A^* B + \bar{c} \delta A) \quad . \quad (3.1.10)$$

We notice that the integral over  $\phi'$  yields  $\delta(\Delta_A^{(0)} \lambda')$ . Since we assume that  $\Delta_A^{(0)}$  does not have any zero-modes then this restricts  $\lambda'$  to be zero. As a result of this, the only appearance of  $\chi$  left is in the action. Integrating over this field we find  $\delta(d_A \psi - \star d_A \eta)$ . Now, the integrability condition for this restriction is  $[F, \psi] = \nabla_A^{(0)} \eta$ . However, as the integral over  $B$  can be seen to enforce  $F = 0$ , we find that  $\eta = 0$ , hence  $d_A \psi = 0$ . This means that all  $\psi$ 's in the path integral are now restricted to be zero-modes. For all but the first term in  $\Lambda_{\mu\nu}$ , the  $\lambda$  integration can be performed and it restricts each  $\phi$  in the  $\mathcal{O}$  to be replaced by an expression (see below) which depends on two  $\psi$  zero-modes. This then means that the path integrals involving each of the last three terms in  $\Lambda_{\mu\nu}$  is saturated by  $\psi$  zero-modes due to the presence of

$\mathcal{O}$ . Thus, we see that the extra  $\oint_\gamma \psi$  due to the Wilson loop makes those expressions vanish. We are then left with the first contribution for  $\Lambda_{\mu\nu}$ . If  $\mathcal{O}$  depends on  $\phi$  this will not be zero. Thus we deduce that  $\frac{\delta \mathcal{E}(R, \gamma, \mathcal{C})}{\delta g^{\mu\nu}} = 0$ , in general, only if the  $\mathcal{O}$  does not depend on  $\phi$ ; otherwise, the only restriction on  $\mathcal{O}$  is that it saturates the number of  $\psi$  zero-modes. Additionally, the result will not be altered if we included more than one Wilson loop in  $\mathcal{E}$ . Thus we conclude that the correlation functions of those  $\hat{\mathcal{O}}$  operators whose ancestors –  $\mathcal{O}$  – saturated the number of fermion zero-modes and are independent of  $\phi$ , with Wilson loops is independent of the background metric.

Observables in the BF-gauge theory which depend on  $B$  have been constructed in the literature [16, 7]. An immediate observation is that if we compute correlation function of quantities which depend on  $B$  then the path integral is not restricted to  $\mathcal{N}$ . This invalidates the proof above. However, if we restrict to  $M = \Sigma \times R$  ( i.e canonical quantization ), then there will be only dependence on  $B|_\Sigma$  in the observables and the restriction to  $F|_\Sigma = 0$  survives. In this case correlation functions involving  $A$ ,  $B$  and  $\hat{\mathcal{O}}(\psi)$  are gauge invariant and metric independent.

Haven given formal expressions for physical correlation functions in BF-gauge theories, we would now like to trace our steps back to the analysis at the beginning of this section and see how it might arise directly from super-BF gauge theories. Let us choose quantum gravity on a genus three handle body as a specific theory; thus,  $g = 3$  and  $G = SO(2, 1)$ . Six  $\psi$  zero modes are needed so we pick three homology 2-cycles which we label as  $\Gamma_i$ . Then we compute the correlation function  $\left\langle \prod_{i=1}^3 \int_{\Gamma_i} \text{Tr}(\psi \wedge \psi) \right\rangle_{SBF}$ , in the super-BF gauge theory also with  $g = 3$  and  $G = SO(2, 1)$ . After integrating out the  $Y$ -fields, we obtain<sup>3</sup>

$$\left\langle \prod_{i=1}^3 \int_{\Gamma_i} \text{Tr}(\psi \wedge \psi) \right\rangle_{SBF} = (Z_{SBF})^{-1} \int [d\mu]_{BF, \alpha_1 \dots \alpha_6} e^{-(S_{BF} + S_{BF, gf})} \hat{\mathcal{O}}(A) \quad , \quad (3.1.11)$$

where

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<sup>3</sup>In general, the path integrations over the bosonic zero-modes are understood to drop out due to the division by  $Z_{SBF}$  in the expressions for the correlation densities.

$$\begin{aligned}
\hat{\mathcal{O}}^{\alpha_1 \dots \alpha_6}(A) = & T(A) \int_{\Gamma_1} Tr(\Upsilon^{\alpha_1}(A) \wedge \Upsilon^{\alpha_2}(A)) \times \\
& \times \int_{\Gamma_2} Tr(\Upsilon^{\alpha_3}(A) \wedge \Upsilon^{\alpha_4}(A)) \times \\
& \times \int_{\Gamma_3} Tr(\Upsilon^{\alpha_5}(A) \wedge \Upsilon^{\alpha_6}(A)) \ . \quad (3.1.12)
\end{aligned}$$

Here, the  $\Upsilon^{\alpha_i}(A)$  form a six-dimensional basis for  $H^1(M, G)$ . By  $[d\mu]_{BF, \alpha_1 \dots \alpha_6}$  we mean  $[d\mu]_{BF}$  with the functional measure over flat connections,  $A^{(0)}$  given by  $[dA_{\alpha_1}^{(0)}] \dots [dA_{\alpha_6}^{(0)}]$ . The  $A_{\alpha_i}^{(0)}$  and  $\Upsilon^{\alpha_i}(A)$  are chosen to form a canonical basis for  $T^*\mathcal{M}$  as in ref. [17]. As this expression was derived directly from the super- $BF$  theory, the result is independent of the choice of basis for the fermionic zero-modes. The quantity  $T(A)$  arises from the non-zero mode integration in  $[d\mu]_S$ . The remaining functional integral has support only on flat connections, hence  $T(A)$  is ostensibly the Ray-Singer (R-S) torsion [18]. We then identify the  $\Upsilon(A)$  as  $\frac{\partial A}{\partial m}$ . Notice that in our analysis of the BF-gauge theory at the beginning of this section, it was not evident that the R-S torsion appears as part of the observable's definition.

Now we realize that

$$\left\langle \prod_{i=1}^3 \int_{\Gamma_i} Tr(\psi \wedge \psi) \right\rangle_{SBF} = \langle \hat{\mathcal{O}}(A) \rangle_{BF} \ . \quad (3.1.13)$$

Then interpreting  $\hat{\mathcal{O}}(A)$  as a correlation density in the BF-gauge theory we continue the computation to find

$$\langle \hat{\mathcal{O}}(A) \rangle = (Z_{SBF})^{-1} \int_{\mathcal{N}} \prod_{i=1}^3 \int_{\Gamma_i} Tr(\Upsilon(A^{(0)}) \wedge \Upsilon(A^{(0)})) \ , \quad (3.1.14)$$

where  $\Upsilon(A^{(0)})$  is a form on the space,  $\mathcal{N}$ , of connections. The functional integral,  $\int_{\mathcal{N}}$  over  $\mathcal{N}$  is done with a wedge product of the  $\Upsilon$ 's, on that space, understood.

As a second example, we construct a correlation density in quantum gravity which is considerably less obvious in the  $BF$  theory than the prior example. We start with  $\int_{\gamma} Tr(\phi\psi)$ , here  $\gamma$  is a one-cycle. It carries ghost number three. Thus we construct a correlation density in quantum gravity on a genus  $g \geq 2$  handle-body given as

$$\hat{\mathcal{O}}(A; \gamma_i) = (Z_{SBF})^{-1} \int [d\mu]_S e^{-S_S} \prod_i \int_{\gamma_i} Tr(\phi\psi) \ . \quad (3.1.15)$$

Integrating over  $\lambda$  we find that at the expense of a factor  $\det^{-1}(\Delta_A^{(0)})$ , we should replace  $\phi(x)$  by  $-\int_{M_y} G_A(x, y)[\psi(y), \star\psi(y)]$ , where  $G_A$  is the Greens' function of the scalar covariant laplacian. Then functionally integrating over  $\psi$  we obtain

$$\begin{aligned}\hat{\mathcal{O}}(A, \gamma_i) &= -(Z_{SBF})^{-1}T(A) \times \\ &\times \prod_i \left\{ \int_{\gamma_i} Tr \left\{ \int_{M_y} G_A(x_i, y) [\Upsilon(A(y)), \star\Upsilon(A(y))] \Upsilon(A(x_i)) \right\} \right\}\end{aligned}\tag{3.1.16}$$

to be another correlation density in quantum gravity. In this expression, the  $\Upsilon$ 's appear anti-symmetrized as in (3.1.12).

Concluding this sub-section, we note one more point about the correlation densities we have been writing down. Unlike observables, our expressions are, in addition to being non-local in the BF-gauge theory, given in terms of path integrals. These functional integrals are best computed in perturbation theory. However, by invoking BRST theorems we were able to obtain some expressions non-perturbatively, in the above. It is safe to say that one lesson we have learned from this sub-section is that for diffeomorphism invariant theories, quantum gravity in particular, we must enlarge our scope of what an observable is. Here, we have used the geometry of the universal bundle and more directly the de Rham complex on moduli space to guide us. Presumably, this direction is worth a try in four dimensions also. We will turn to the latter in the next section. However, before that, we would like to discuss some even more profitable results; namely, expressions for wavefunctionals based on the universal bundle geometry.

## 3.2 Wavefunctionals

The physical Hilbert space of a super-BF gauge theory consists of  $L^2$ -functions on the moduli space,  $\mathcal{M}$ , of flat connections. In principle, quantization of this field theory is then reduced to quantum mechanics on  $\mathcal{M}$ . However, the pragmatism of such a program is limited as, a priori, it becomes unwieldy to pull such wavefunctions back into wavefunctional of the connection. In this sub-section we will demonstrate how this problem may be obviated. To be precise, we will write down expressions for the functionals

of the connections on the  $G$ -bundle which are annihilated by the constraints of the theory.

Correlation functions of observables in TQFT's are equal to the integral over moduli space of a top form on that space [17]. Typically, such a top form is wedge product of forms of lesser degree:

$$\left\langle \prod_i^d \mathcal{O}_i \right\rangle = \int_{\mathcal{M}} \Psi \ , \quad \Psi = f_1 \wedge f_2 \wedge \cdots \wedge f_d \ , \quad (3.2.1)$$

where the forms,  $f_i$ , are obtained after integrating over the non-zero modes and fermionic zero-modes in the path integral. Now, let us assume that a metric exists on  $\mathcal{M}$  so that we can define the Hodge dual map which we denote by the tilde symbol. Then  $\tilde{\Psi}$  is a scalar function on moduli space. Let us now give representations for  $\Psi$ . All we seek is  $\Psi$ 's which are gauge invariant and have support only on flat connections,  $\omega$ , on  $\Sigma_g$ :  $\Psi[\omega]$ .

Clearly [19], a delta function,  $\delta(F)$ , where  $F$  is the curvature of a  $G$ -bundle over  $\Sigma_g$  satisfies our criteria. However, as it is highly unlikely to be normalizable. Regardless, we realize that it might be worthwhile to look at diffeomorphism invariant theories on  $\Sigma_g$  which are defined in a flat connection background. Considering the two-dimensional BF-gauge theory we see that the analog of  $B$ ,  $\varphi$ , is an  $ad(G)$ -valued zero form and the action is  $S_{BF}^{2D} = \int_{\Sigma_g} Tr(\varphi F)$ . The delta function arises from the integral over  $\varphi$ . If we constructed the analog of  $S_{SBF}$ , we would find that it shares many of the terms which appear in the three dimensional action. However, here the analog of  $\chi$  (we will call this  $\xi$  below) is a zero-form. What is more, there are no primed fields due to the degree of  $\xi$ . Considering this, we introduce the functional  $\int_{\Sigma_g} Tr(\xi d_\omega \psi)$ , for  $ad(G)$ -valued, Grassmann-odd zero- ( $\xi$ ) and one- ( $\psi$ ) forms defined in a flat connection background,  $\omega$ . It is invariant under the local symmetry,  $\delta\psi = \epsilon d_\omega \phi$  and upon gauge fixing it we obtain the quantum functional

$$S_\omega = \int_{\Sigma_g} Tr(\xi d_\omega \psi - \eta d_\omega^* \psi - \lambda^* \triangle_\omega \phi) \ . \quad (3.2.2)$$

The partition function for this action is metric independent as the part of  $S_\omega$  which is metric dependent is exact with respect to the BRST charge for the gauge fixing of the symmetry just discussed. Furthermore, it is simple enough to compute exactly and is found to be equal to the Ray-Singer torsion



of the  $G$ -bundle with flat connection,  $\omega$ . In fact, the action  $S_\omega$  is recognized as the action for a two-dimensional Grassmann-odd BF field theory in a flat connection background. As was the case with the super-BF gauge theory, the correlation functions of quantities such as  $\frac{1}{2}Tr(\phi^2(x))$ , etc., are topological invariants. This is seen to be due to the transformation given by the BRST charge:  $\{Q, \psi\} = d_\omega \psi$ . The partition function has support only on solutions of those  $\psi$  which are in  $\ker(d_\omega)$ . Hence, they span the cotangent space of  $\mathcal{M}(\Sigma_g)$  whose dimension is  $(2g - 2) \dim(G)$ .

As before, let us focus on three dimensional quantum gravity taking  $G = SO(2, 1)$ . Our first example of a wavefunctional is found by taking the  $(3g - 3)$  times product of  $\int_{\Gamma_i} Tr(\psi \wedge \psi)$  where the  $\Gamma_i$  are homology 2-cycles in  $\Sigma_g$ :

$$\Psi[\omega] = \int [d\xi][d\psi][d\eta][d\lambda][d\phi] e^{-S_\omega} \prod_{i=1}^{(3g-3)} \int_{\Gamma_i} Tr(\psi \wedge \psi) \quad , \quad (3.2.3)$$

defined over the two-dimensional super-BF theory. The generic form of the wavefunctionals obtained by this construction is

$$\begin{aligned} \Psi_{\vec{n}}[\omega] &= \int [d\xi][d\psi][d\eta][d\lambda][d\phi] e^{-S_\omega} \prod_{i=1}^{n_4} Tr(\phi^2(x_i)) \times \\ &\times \prod_{j=1}^{n_3} \oint_{\gamma_j} Tr(\phi\psi) \prod_{k=1}^{n_2} \int_{\Gamma_k} Tr(\psi \wedge \psi) \quad , \end{aligned} \quad (3.2.4)$$

subject to the condition  $4n_4 + 3n_3 + 2n_2 = \dim \mathcal{M}(\Sigma_g, G)$ . If  $\omega$  is not an irreducible connection, then there are no  $\phi$  zero-modes and the only non-zero  $\Psi_{\vec{n}}[\omega]$  are those for which  $n_4 = n_3 = 0$ .

In the preceding, we have not used the full power of the two-dimensional super-BF gauge theory. As a matter of fact, we did not use it at all. The transition from  $S_\omega$  to the super-BF gauge theory on  $\Sigma_g$  is straightforward. Its action is

$$S_{SBF}^{2D} = \int_{\Sigma_g} Tr(\varphi F) - S_\omega + \int_{\Sigma_g} Tr(\lambda[\psi, {}^*\psi]) \quad , \quad (3.2.5)$$

where  $\varphi$  is a zero-form which imposes the flat connection condition on  $\omega$  and the rest of the action is reminiscent of the three dimensional theory but without the primed fields. Unlike the pure  $S_\omega$  theory, the absence of  $\phi$  zero-modes does not imply  $\phi = 0$ , but  $\phi(x) = - \int_{\Sigma_{g,y}} G_A(x, y)[\psi(y), {}^*\psi(y)]$  as

we saw in the previous sub-section. Thus, more wavefunctionals result from this theory. They are of the same form as  $\Psi_{\vec{n}}$  except that the functional measure must be enlarged to include all the fields in  $S_{SBF}^{2D}$ . Additionally,  $S_{\omega}$  is replaced by  $S_{SBF}^{2D}$ . We find the general form of these wavefunctionals to be

$$\begin{aligned} \Psi_{\vec{n}}^S[\omega] &= (-)^{n_3} \prod_i^{n_4} Tr((\int_{\Sigma_{g,y}} G_{\omega}(x_i, y)[q(y), {}^*q(y)])^2) \times \\ &\times \prod_j^{n_3} \oint_{\gamma_j} Tr(\int_{\Sigma_{g,y}} G_{\omega}(x_j, y)[q(y), {}^*q(y)]q(x_j)) \times \\ &\times \prod_k^{n_2} \int_{\Gamma_k} Tr(q \wedge q) , \end{aligned} \quad (3.2.6)$$

again with  $n_4 + n_3 + n_2 = \dim(\mathcal{M}(\Sigma_g, G))$  and where the  $q(\omega)$  form a  $(2g - 2) \dim(G)$  dimensional basis for  $H^1(\Sigma_g, G)$ . As was the case in eqn. (3.1.12), the  $q$ 's appear in a totally anti-symmetric combination.

Our philosophy thus far has been to identify a Riemann surface (which is homeomorphic to the hypersurface of the foliated three-dimensional BF-gauge theory) and construct a servant partition function<sup>4</sup> for fields in a background which solves the constraints of the three-dimensional BF-gauge theory. Having done this we then identified operators which yield diffeomorphism invariant observables in the two-dimensional topological “theory”. We assume that we can solve the equation which defines the constraints (as though they were classical equations) and parametrize them by the coordinates on moduli space. For example, the  $\omega$  which defines the background above is really  $\omega(x; m)$ . This means that the wavefunctionals are not simply defined at one point in moduli space, but rather on all of  $\mathcal{M}(\Sigma_g, G)$ . We advocate this as a very robust approach to constructing quantum gravity wavefunctionals as (1) we need only solve the constraints classically and (2), thanks to our experience with TFT's it is rather straightforward to at least formally construct the servant partition functions and correlation functions in such a parametrized background.

Now, the fact that, in the previous sub-section, we were successful in formally constructing correlation densities leads to another possible approach to constructing wavefunctionals. If those correlation densities can be written,

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<sup>4</sup>We will call these servant partition functions to distinguish them from the partition functions of the theories we are constructing the wavefunctionals for.

as operators, as  $\hat{\mathcal{O}} = \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}}$  for some operator  $\hat{\mathcal{O}}$  and adjoint,  $\dagger$ , then we would have  $\langle \hat{\mathcal{O}} \rangle = \langle 0 | \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}} | 0 \rangle$ . Interpreting this as the norm of a state  $\hat{\mathcal{O}}|0\rangle$ , it is suggestive that the wavefunctional of such a state may be formed from the path integral expression for the correlation density. The manner in which we see this state arising is analogous to sewing in the super-BF gauge theory. Hence, we expect to be able to form the corresponding wavefunctional by surgery in the super-BF gauge theory. Although we delay detailed investigation of such an approach until a future publication, we would like to point out here that normalized wavefunctionals are expected. In the rest of this sub-section, we will show how to construct wavefunctionals from a super-BF gauge theory on a three manifold whose boundary is  $\Sigma_g$ .

We start with a super-BF gauge theory for a  $G$ -bundle over a three-dimensional manifold  $M$  with boundary  $\Sigma_g$ . Then we insert the pertinent operators as was done in the previous sub-section. Having done this, we choose a polarization (for which the fields in the geometrical sector appear as “position” variables) and perform all functional integrals with appropriate boundary conditions. This gives a wavefunctional for the super-BF gauge theory which is annihilated by the full BRST operator. It is also gauge and diffeomorphism invariant [17, 20]. In general, there may be fields which do not depend on the boundary values of the geometrical sector. Starting with the wavefunctional of the super-BF gauge theory, we integrate over their boundary values. This leads to a functional,  $\Psi[\omega, \varpi]$ , where  $\omega$  is a flat connection on  $\Sigma_g$ ,  $\varpi$  denotes a zero-mode of  $\psi$  and is a solution of  $d_\omega \varpi = 0$  on  $\Sigma_g$ , and we have replaced the boundary value of  $\phi$  with the appropriate expression in terms of  $\omega$  and  $\varpi$ .  $\Psi$  also depends on the boundary values of  $c$  and  $c'$ ; however, for notational simplicity they will be omitted.

Focusing our attention on genus- $g$  handle-bodies,  $\partial M = \Sigma_g$ , we compute the correlation functions for the topological invariants fixing the boundary value of the connection to be a flat connection on  $\Sigma_g$ . This can be done by inserting a delta function,  $\delta(U_I(A) - g_I(\omega))$  for each longitude,  $l_I$  in  $M$ . Here,  $U_I(A)$  is the holonomy of the connection along the longitude,  $l_I$ , and  $g_I(\omega)$  is the holonomy of a parametrized flat connection on the cycle,  $b_I$ , on  $\Sigma_g$  which (in the handlebody) is homotopic to  $l_I$ . For the  $\psi$  field, we insert delta functions  $\delta(\oint_{l_I} \psi - \oint_{b_I} \varpi)$  where  $l_I$  and  $b_I$  are as before. Consequently, we

arrive at our generic ansatz for wavefunctionals of super-BF gauge theories:

$$\begin{aligned} \Psi[\omega, \varpi] = & \int [d\mu]_{SBF} \prod_{I=1}^{(g-1)\dim(G)} \delta(U_I(A) - g_I(\omega)) \prod_{J=1}^{(g-1)\dim(G)} \delta(\oint_{l_J} \psi - \oint_{b_J} \varpi) \times \\ & \times \prod_i \mathcal{O}_i(\phi, \psi, F; \mathcal{C}_i) e^{-S_{SBF}} . \end{aligned} \quad (3.2.7)$$

In this expression, the product of polynomials,  $\prod_i \mathcal{O}_i$  is such that it saturates the number of  $\psi$  zero-modes.

Now we must project out  $\varpi$ . This we do by treating  $\Psi[\omega, \varpi]$  as a superfield and obtaining the wavefunctional of  $\omega$  as a component via superfield projection. Choose a  $(2g-2)\dim(G)$ -dimensional basis,  $q_\alpha(\omega)$ , for  $H^1(\Sigma_g, G)$  and expand the one-form field,  $\varpi$ , in it as  $\varpi \equiv \sum_\alpha \theta^\alpha q_\alpha$ , where the Grassmann-odd coefficients  $\theta^\alpha$  are the quantum mechanical oscillators. As the wavefunctional depends on  $H^1(M, G)$  only  $(g-1)\dim(G)$  of the  $\theta^\alpha$  will be non-zero. Then,  $\Psi$  is formally  $\Psi[\omega, \theta]$  and we write,

$$\Psi_{\alpha_1, \dots, \alpha_n}[\omega] = \frac{\partial^n}{\partial \theta^{\alpha_n} \dots \partial \theta^{\alpha_1}} \Psi[\omega, \theta] \Big| , \quad (3.2.8)$$

where the slash means setting  $\theta = 0$  after differentiating. Each  $\Psi_{\alpha_1, \dots, \alpha_n}$  for  $n = 1, \dots, (g-1)\dim(G)$ , is a wavefunctional in the BF-gauge theory in that it satisfies the constraints of the latter theory. We adopt this procedure as it is closest to the sewing/surgery procedure, is of geometrical significance (see below) and it incorporates the two naive guesses: setting  $\varpi = 0$  or integrating out  $\varpi$ .

The closest analogy of these  $\Psi_{\alpha_1, \dots, \alpha_n}[\omega]$  is to Hartle-Hawking wavefunctionals [21]. Notice that unlike the previous wavefunctionals,  $\Psi_n^S$ , in eqn. (3.2.6), which are analogous to canonical wavefunctionals, these are functionals only of half of the flat connections on  $\Sigma_g$ . This is due to the fact that the meridians of the handlebody are contractible in  $M$ .

Based on our discussion at the beginning of this sub-section, we realize that the  $\Psi_{\alpha_1, \dots, \alpha_n}$  are  $n$ -forms on moduli space. This returns us to our earlier discussion of the wavefunctionals of quantum gravity as being  $L^2$ -functions on moduli space. In writing down the  $\Psi_{\alpha_1, \dots, \alpha_n}$ , we have given formal expressions for the pertinent functions on the moduli space in terms of the physical variable in the problem; namely, the connection.

The question remains which of them is normalizable. Although we do not have much to say about this question in this work, we would like to bring

to the fore a possible strategy for normalizing wavefunctionals constructed in this way. Any closed piecewise linear three dimensional manifold,  $N$ , may be formed via the Heegaard splitting,  $N = M_1 \cup_h M_2$ , where  $M_1$  and  $M_2$  are two handlebodies whose boundaries are homeomorphic (with map,  $h$ ) to each other [22]. Making use of this, we view [12] the norm of  $\Psi[\omega, \theta]$  to be a functional integral on  $N$ ,  $\Psi[\omega, \theta]$  itself to be the functional integral on  $M_1$  and its adjoint to be the functional integral on  $M_2$ . Reversing, we start with the functional integral for the correlation functions of polynomial invariants on  $N$ , perform surgery and then identify that component of  $\Psi$  which appears in the form  $\Psi[\omega]^\dagger \Psi[\omega]$  after integrating  $[d\mu]_S$ . Obtaining the adjoint ( $^\dagger$ ) is interpreted as arising from the surgery/sewing process.

## 4 Application to 4D Quantum Gravity

In this section, we sketch how the above construction can be realized in four dimension quantum gravity. The main ingredient that we have to supply is a TQFT whose action starts off as an Einstein-Hilbert action, or rather when some of the fields are put to zero one gets the usual action for gravity. Generally, we expect that there is more than one such action. Unlike the three dimensional case where quantum gravity was already defined over a finite dimensional phase space (flat connections modulo gauge transformation), in four dimension there are propagating fields and we can then try projecting the theory onto many different moduli spaces. In a way, the Einstein-Hilbert action is a good example for this construction, Whereas it is non-renormalizable, the topological projection takes us to a renormalizable theory in which to do calculations on physical observables for quantum gravity. In addition, we can construct formal, diffeomorphism invariant expressions<sup>5</sup> without integration over the whole manifold.

Our first task is to select a TQFT. The logical candidates are topological gravity theories. Four dimensional topological gravity theories for which the pure metric part of the action is given by the square of the Weyl tensor, not the Einstein-Hilbert action, are known [23]. As mentioned above, we seek a

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<sup>5</sup>We remind the reader that in constructing observables, we work in covariant – not canonical – quantization.

topological gravity theory whose pure metric action is the Einstein-Hilbert action. Now, TQFT's may be obtained from supersymmetric theories via twisting [17]. The fact that the four dimensional gravity theories which were first constructed were conformal is apparently correlated with the fact that N=2 supergravity in four dimensions has this feature. A Poincaré supergravity theory was proposed sometime ago by de Wit<sup>6</sup> [24]. Thus we expect that a twisted version of this should exist as a topological gravity theory.

In ref. [25], a topological gravity theory with Einstein-Hilbert action as is pure metric part was obtained by twisting a N=2 supergravity theory. Here, we will simply use the results of this work. The twisting procedure defined a Lorentz scalar, Grassmann-odd (BRST) charge,  $Q$  which is nilpotent. As it turns out this topological gravity theory is seen to be the TQFT for the projection of the spin-connection form (in the second order formulation) to be self-dual:

$$w^{-ab}(e) = 0 \ , \quad (4.0.1)$$

where  $a$  etc. are Lorentz indices.

The observables are constructed from the cohomology of  $Q$ . After some re-definitions of the fields one ends up with a BRST charge whose action upon the geometrical sector of the theory is

$$\begin{aligned} Q : e^a &\rightarrow \psi^a - \mathcal{D}\epsilon^a + \epsilon^{ab} \wedge e_b \ , \\ Q : w^{ab} &\rightarrow \chi^{ab} - \mathcal{D}\epsilon^{ab} \ , \\ Q : \psi^a &\rightarrow -\mathcal{D}\phi^a - \eta^{ab} \wedge e_b - \chi^{ab} \wedge \epsilon_b + \epsilon^{ab} \wedge \psi_b \ , \\ Q : \phi^a &\rightarrow \epsilon^{ab} \wedge \phi_b \rightarrow \eta^{ab} \wedge \epsilon_b \ , \\ Q : \chi^{ab} &\rightarrow -\mathcal{D}\eta^{ab} + \epsilon^{ac} \wedge \chi_c^b - \chi^{ac} \wedge \epsilon_c^b \ , \\ Q : \eta^{ab} &\rightarrow \epsilon^{ac} \wedge \eta_c^b - \eta^{ac} \wedge \epsilon_c^b \ , \\ Q : \epsilon^a &\rightarrow \phi^a + \epsilon^{ab} \wedge \epsilon_b \ , \\ Q : \epsilon^{ab} &\rightarrow \eta^{ab} + \epsilon_c^a \wedge \epsilon^{cb} \ , \end{aligned} \quad (4.0.2)$$

where  $\epsilon^{ab}$  and  $\epsilon^a$  are the ghosts for Lorentz and diffeomorphism symmetries, respectively. As was mentioned above all this is in second order formalism. Although in the BRST transformations all the fields look independent, this is not the case. However, according to [25], these transformations are consistent

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<sup>6</sup>We thank S. J. Gates, Jr. for bring this to our attention.

with the conditions of the second order formalism :  $w^{ab} \wedge e_b = de^a$  ,  $\chi^{ab} \wedge e_b = -\mathcal{D}\psi^a - R^{ab} \wedge \epsilon_b$  , etc...

The cohomology is constructed exactly as in Donaldson-Witten theory [7] and observables are found. For example,

$$\begin{aligned}\mathcal{O}^{(4)} &= \frac{1}{2}Tr(\eta^2) \ , \\ \mathcal{O}^{(3)} &= \int_{\gamma} Tr(\eta\chi) \ , \\ \mathcal{O}^{(2)} &= \int_{\Gamma} Tr(\eta R + \frac{1}{2}\chi \wedge \chi) \ , \\ \mathcal{O}^{(0)} &= \int_M Tr(R \wedge R) \ .\end{aligned}\tag{4.0.3}$$

The geometrical meaning of each field is in pure analogy with those in Donaldson-Witten theory. As in the three dimensional case, the correlation functions of these observables become, after integration over the non-zero-mode parts of the fields, functions of the zero modes of  $\chi^{ab}$  and the veirbein,  $e^a$ . The zero-modes of  $\chi$  are a basis for the cotangent space to the moduli space of  $w^{-ab}(e) = 0$ .

Now that we have described the results of [25] we would like to indicate how the constructions of the previous sections can be applied here. To construct the observables we insert, in the path integral a combination of the TQFT observables which saturates the  $\chi$  zero-modes and then integrate over all the fields except  $e, \epsilon^a$  and  $\epsilon^{ab}$ . This will result in a non-local expression in terms of the veirbein whose expectation value is a topological invariant of space time. In this way, we obtain  $\hat{\mathcal{O}}(e)$  from the  $\mathcal{O}^{(k)}$  above.

The wavefunctional construction is also very similar. However, at present, we can only construct the Hartle-Hawking type wavefunctionals, as unlike the 3D case, we do not have the corresponding servant action. Defining the action to be on an four dimensional manifold,  $M$ , with boundary  $\partial M = \Sigma$ , we obtain wavefunctionals by following the same steps we took in the three dimensional case. This results in a functional  $\Psi[e|_{\Sigma}]$ . The proof that the wavefunctional so constructed satisfies the constraints of quantum gravity is now the same as that of Hartle-Hawking [21]. We differ the exact results, in particular, a presentation of the normalization procedure, to a future work [12].

## 5 Conclusions

In this work we have indicated a possible way of using TQFT's to construct wavefunctionals and physical correlation functions in three and four dimensional quantum gravity. We gave explicit results in the three dimension case and laid the building blocks for the construction in four dimensions. Along the way we have shown that in quantum gravity, there are functions of the fields whose vacuum expectation values are not only diffeomorphism invariants of spacetime, but also of geometrical significance on moduli space. A possible definition of an inner product was mentioned and will be elaborated in [12]. This work also indicates that it might be useful to consider quantum gravity in a larger geometrical setting than usual.

In concluding, it is tempting to speculate that pursuit along the lines advocated in this work may lead to possible field theoretic relations between intersection numbers on a manifold and the intersection numbers on the moduli space of field configurations for sections of bundles over that manifold. In particular, we know that the Wilson loop observables in BF-gauge theories [26] and the loop observables [2] construct knot invariants. Well, we have found the projections of Donaldson-like polynomial invariants into three and four dimensional quantum gravity. From a purely field theoretical point-of-view, we then expect to find a relation between these two sets of operators.

## Acknowledgements

We thank M. Blau for a critical reading of the manuscript.



## Appendices

### A 3D BF Gauge Theory

Recall [7] that the actions of BF-gauge theories are topological and of the form

$$S_{BF} = \int_M \text{Tr}(B \wedge F) , \quad (\text{A.1})$$

where  $B$  is an  $ad(G)$ -valued  $(n-2)$ -form on the oriented, closed  $n$ -manifold,  $M$  and  $F$  is the curvature of the  $G$ -bundle whose connection is  $A$ . Path integrals for these theories with the insertion of any operators except those composed of  $B$  have support only on flat connections. The wavefunctionals for these theories reduce to  $L^2$ -functions on the moduli space,  $\mathcal{M}$ , of flat connections. For purposes of path integral quantization, the partition function of the BF-gauge theory is

$$Z_{BF} = \int [dA][dB][d\bar{c}][dc][d\bar{c}'][dc'] [db][db'] e^{-[S_{BF} + S_{BF,gf}]} , \quad (\text{A.2})$$

where

$$S_{BF,gf} = \int_M \text{Tr}(bd^*A + b'd_A^*B - \bar{c}d^*d_{Ac} - \bar{c}'d^*d_{Ac'}) . \quad (\text{A.3})$$

The last action represents the projection of the connection into the Lorentz gauge and the removal of the covariantly exact part of  $B$ ; all done by means of the symmetries of the BF-gauge theory. In this gauge fixing, the  $c(c')$  and  $\bar{c}(\bar{c}')$  fields are the zero-form, anti-commuting ghosts and anti-ghosts for the  $A(B)$  projections, respectively.

Canonical quantization on  $M = \Sigma \times \mathbb{R}$  immediately leads to the constraints [27],

$$^*d_A B \approx 0 , \quad ^*F \approx 0 , \quad (\text{A.4})$$

where the Hodge dual here is defined on  $\Sigma$  and is induced from that on  $M$ . The first of these constraints is Gauss's law enforcing the gauge invariance of physical states and the second requires that these states have support only on flat connections. In this special case of 2+1 dimensional quantum gravity, it can be shown [14, 5] that on physical states  $\text{Diff}(\Sigma)$  is equivalent to these constraints.

## B 3D Super-BF Gauge Theory

The action for super-BF gauge theory [7, 28] is

$$\begin{aligned}
S_{SBF} = \int_M \text{Tr} \{ & B \wedge F - \chi \wedge d_A \psi \\
& + \eta d_A^* \psi + \lambda^* \Delta_A \phi + \lambda [\psi, {}^* \psi] \\
& + \eta' d_A^* \chi + \lambda'^* \Delta_A \phi' + \lambda' [\psi, {}^* \chi] \} . \quad (B.1)
\end{aligned}$$

All fields are  $ad(G)$  valued and their form degree, Grassmann-parity and fermion/ghost number are listed in the following table:

FIELD	DEGREE	G-PARITY	GHOST #
$B$	1	even	0
$A$	1	even	0
$\chi$	2	odd	-1
$\psi$	1	odd	1
$\eta$	0	odd	-1
$\eta'$	0	odd	1
$\lambda$	0	even	-2
$\phi$	0	even	2
$\lambda'$	0	even	0
$\phi'$	0	even	0

Placing this set of fields in the context of section 2, the BF-gauge theory is the GFT and super-BF gauge theory is the TQFT. The sets of fields are represented by  $X = (B, A)$  with  $Y$  being the rest of the fields in this table.

The (Yang-Mills) gauge invariant action (B.1) may be obtained by starting with the zero lagrangian and gauge fixing the topological symmetry  $\delta A = \epsilon \psi$  to the flat connection condition,  $F = 0$ . It is invariant under the horizontal BRST transformations

$$\begin{aligned}
Q^H : A &\rightarrow \psi , & Q^H : \psi &\rightarrow d_A \phi , \\
Q^H : \chi &\rightarrow B + d_A \phi' , & Q^H : B &\rightarrow [\chi, \phi] + [\phi', \psi] , \\
Q^H : \lambda &\rightarrow \eta , & Q^H : \eta &\rightarrow [\lambda, \phi] , \\
Q^H : \lambda' &\rightarrow \eta' , & Q^H : \eta' &\rightarrow [\lambda', \phi] . \quad (B.2)
\end{aligned}$$

Additionally, it is invariant under the one-form symmetry  $\delta B = d_A \Lambda$ . The gauge fixing of these symmetries introduces the usual ghost “kinetic” terms plus some new terms which involve Yukawa-like couplings with  $\psi$  and  $B$ . We will return to these later. It is worthwhile to note that the action,  $S_{SBF}$  may be written as the action of a BF-gauge theory plus “supersymmetric” completion term as<sup>7</sup>:

$$S_{SBF} = S_{BF} + S_S . \quad (\text{B.3})$$

The partition function for super-BF gauge theory is (see appendix D for our notation)

$$Z(M) = \int [d\mu]_{SBF} e^{-S_{SBF}} . \quad (\text{B.4})$$

Integrating over  $B$  we see that this partition function has support only on flat connections as is the case with the BF-gauge theories. However,  $\psi$  and  $\phi$  have made their appearances. The observables [7] of this theory are elements of the  $Q^H$ -equivariant cohomology and are maps from  $H_*(M)$  to  $H^*(\mathcal{M})$ . For rank two groups they are constructed as polynomials of the following homology cycle integrals:

$$\begin{aligned} \mathcal{O}^{(4)} &= \frac{1}{2} \text{Tr}(\phi^2) , \\ \mathcal{O}^{(3)} &= \int_{\gamma} \text{Tr}(\phi\psi) , \\ \mathcal{O}^{(2)} &= \int_{\Gamma} \text{Tr}(\phi F + \frac{1}{2} \psi \wedge \psi) , \\ \mathcal{O}^{(1)} &= \int_M \text{Tr}(\psi \wedge F) . \end{aligned} \quad (\text{B.5})$$

In these expressions, the index  $(k)$  represents the fact that  $\mathcal{O}^{(k)}$  is a  $k$ -form on moduli space or carries ghost number  $k$  in the BRST language. These are the three dimensional analogs of the Donaldson invariants which may be constructed in four dimensional topological Yang-Mills theory [17].

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<sup>7</sup>We use  $S_S$  heavily in the text.

## C Special Topology for BF: $\Sigma_g \times S^1$

In this appendix, we would like to suggest the possible existence of polynomial invariants in the pure BF-gauge theory if the three-dimensional manifold is taken to be the Lens space  $S^2 \times S^1$  or  $\Sigma_g \times S^1$ , in general<sup>8</sup>. In what follows we will assume the temporal gauge. However, this is not completely possible due to the holonomy of the gauge field in the  $S^1$  direction. It is for this reason that our discussion is only suggestive. Nevertheless, see ref. [30] in which an explicit demonstration of the relation between Chern-Simons theory and G/G WZW theory on  $\Sigma_g$  is given.

Expand all of the fields in the harmonics of  $S^1$ ,  $e^{in\theta}$ , (where  $\theta$  is the coordinate on  $S^1$  and  $n$  is an integer) symbolically as  $\Phi(\Sigma_g \times S^1) \equiv \sum_n \Phi_{(n)}(\Sigma_g) e^{in\theta}$ . Then choose the “temporal gauge” so that the connection in the  $S^1$  direction vanishes leading to the action<sup>9</sup>

$$S_{BF} = \int_{\Sigma_g} Tr \left( \sum_n \phi_{(n)} dA_{(-n)} + \sum_{n,m} \phi_{(n)} A_{(m)} \wedge A_{(-n-m)} - i \sum_n n B_{(n)} A_{(-n)} + \sum_n n \bar{c}_{(n)} c_{(-n)} \right) , \quad (C.1)$$

where  $\phi_{(n)}$  are the components of the original  $B$  field in the  $S^1$  direction. Realizing that the  $B_{(n)}$  for  $n \neq 0$  does not appear in a term with derivatives, we integrate it out of the action finding

$$S_{BF} = \int_{\Sigma_g} Tr \left( \phi F + \sum_n n \bar{c}_{(n)} c_{(-n)} \right) , \quad (C.2)$$

where  $\phi \equiv \phi_{(0)}$  and  $F$  is the curvature on  $\Sigma_g$  constructed out of  $A_{(0)}$ . With the exception of the completely decoupled fermionic term, this is the action for two dimensional  $BF$  theory. It has been recently studied quite extensively [29, 20]. Notice that we did not obtain it via compactifying the  $S^1$  direction. That direction simply decouples due to the first order and off-diagonal nature of the gauge theory. Extending the theory to incorporate an equivariant cohomology is possible, however, we will not need this in order to obtain our results.

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<sup>8</sup> Here, as in the text,  $\Sigma_g$  is a genus  $g$  Riemann surface.

<sup>9</sup>In this appendix,  $\phi$  is not the scalar field in the super-BF gauge theory considered in the body of the paper.

From the partition function for this action, it is easy to show that gauge invariant functions of  $\phi$  will be invariant under diffeomorphisms. As  $\mathcal{L}_K\phi = i_K d_A\phi + [\phi, \alpha(K)]$ , we must show that  $\langle d\mathcal{O}(\phi) \rangle = 0$ , where  $\mathcal{O}$  is some gauge invariant function constructed only out of  $\phi$ . This equality follows from the statement that  $d_A\phi$  is obtained by varying the action with respect to the connection, thus its expectation value and/or correlation function with any other functions of  $\phi$  is a total functional derivative on  $\mathcal{A}$ ; hence it vanishes. Another way to see this result is that a symmetry of the action (C.1) exists in which  $\phi$  may be shifted into a  $\bar{c}_n$  (or  $c_n$ ). This symmetry, however, does not affect the  $\phi$  zero-mode (solution of  $d_A\phi = 0$ ) as it does not appear in the action. The exponent in definition of  $\mathcal{B}_I$  is  $Q^H$  exact and since it is constructed to be gauge invariant it is also  $Q^H$  closed.

## D Notation

The symbol,  $G$  is used to denote a semi-simple Lie group. The space of gauge connections is written as  $\mathcal{A}$ . Our generic notation for spacetime manifolds is  $M$  while we use the symbol,  $\mathcal{M}(M, G)$  for the moduli space of specific (e.g., flat) connections for the  $G$ -bundle,  $P$ , over  $M$ . The exterior derivative on  $M$  is  $d$  while the covariant exterior derivative with respect to the connection  $A$  is  $d_A$ . Coordinates on  $M$  are written as  $x, y$ , etc. while coordinates on  $\mathcal{M}(M, G)$  denoted by  $m$ . The gauge covariant laplacians on  $k$ -forms are written as  $\Delta_A^{(k)}$ . The genus of a handlebody/Riemann surface is  $g$ . Any metrics which appear explicitly are written with indices or otherwise in an obvious manner. While  $\gamma$  denotes a homology one-cycle,  $\Gamma$  stands for a homology two-cycle. The functional measures used are defined in the following table:

NOTATION	MEASURE	ACTION
$[d\mu]_{BF}$	$[dA][dB][d\bar{c}][dc][d\bar{c}'][dc'][db][db']$	$S_{BF} + S_{BF,gf}$
$[d\mu]_S$	$[d\chi][d\psi][d\eta][d\eta'][d\phi][d\lambda][d\phi'][d\lambda']$	$S_S$
$[d\mu]_{SBF}$	$[d\mu]_{BF}[d\mu]_S$	$S_{SBF}$

All other notations are established in the text. Except note that our field notations for the TQFT's are not the same in three and four dimensions (see section 4).

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